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Mathematical By-products

In his "Science in a Tavern" (University of Wisconsin Press, 1938) Dean Emeritus C. S. Schlicter says: "In my lifetime there have been two developments in the power and authority of mathematics. . . . One of these is the amazing fact that all natural science has become mathematical." So, in the massive evolution of processes making for man's mastery of Nature, mathematics has come to play an essential role, not alone in the various inorganic divisions of Nature's domain, but even in the spheres of organic life. To chemistry, with its organic applications, are lately added such significant fields of research as biometrics, and econometrics—terms which, on their very face, show the fingerprints of *mathematics*.

The utility values of mathematics in uncounted fields of busy human movement are now questioned only by the grossly ignorant. The rate at which mathematical method shall find its application in fields not now using it may, reasonably, be a matter of question, but is unimportant, since time will tell the story.

Nor is the writer presently concerned with the more or less academic question of discipline aspects of mathematical study. He *is* concerned to list on the balance of this page a few of the things that are unquestionably valuable to one's thinking techniques and habits, whether one be technician, engineer, lawyer, business man, or plies a vocation of whatever kind—*things* that are most easily appropriated from the body of mathematics.

In mathematics: (1) Every concept is represented by a written symbol. (2) Every symbol has assigned to it one and only one value, except as in (3). (3) If a symbol has a variable value the mode of the variation is definitely specified. (4) An aggregate of symbols has one and only one value or meaning, except as affected by condition (3). (5) A *succession* of *symbol aggregates* defines one and only one logically deduced value, though such value may be a variable through (3) and (4). (6) The values, or meanings, arrived at in mathematics are assumed to be consistent, or non-contradictory. (7) Concentration of mind, being continuously demanded, early becomes a habit.

S. T. Sanders.

Theorems, Their Converses and Their Extensions

By N. A. Court
University of Oklahoma

A.

1. a. Given two circles (A), (B), their centers of similitude Z, Z' divide their line of centers AB harmonically, and their circle of similitude (ZZ'), i. e., the circle on ZZ' as diameter, is coaxal with them.*

b. Converse theorem. *If a circle is coaxal with two given circles and divides their line of centers harmonically, it coincides with their circle of similitude.*†

The variable circle having for ends of a diameter a pair of points harmonically separated by the centers A, B of the two given circles (A), (B) describes a coaxal system (I) of which A, B are the limiting points. The circle of similitude of (A), (B) belongs to the coaxal system determined by (A) and (B), according to the direct theorem, and obviously belongs to the coaxal system (I). But the two coaxal systems cannot have more than one circle in common, hence the proposition.

The theorem obviously applies to the sphere of similitude of two spheres.‡

2. a. Given three circles (A), (B), (C), the three pairs of centers of similitude X and X', Y and Y', Z and Z' of the three pairs of circles (B) and (C), (C) and (A), (A) and (B) are pairs of opposite vertices of a complete quadrilateral (Q) of which ABC is the diagonal triangle.§

b. Converse theorem. *With the vertices of the diagonal triangle of any complete quadrilateral as centers three circles may be described so that the three pairs of opposite vertices of the quadrilateral will be the pairs of centers of similitude of the three circles taken two -by-two.*

*Nathan Altshiller-Court, *College Geometry*, p. 196, Art. 391. Richmond, Va., 1925. This book will be referred to as *CG*.

†American Mathematical Monthly, Vol. 39, 1932, p. 56, Q. 3477.

‡Cf. Nathan Altshiller-Court, *Modern Pure Solid Geometry*, p. 186, Art. 591. The Macmillan Co., New York, 1935. This book will be referred to as *MPSG*.

§Cf. *CG*., pp. 160, 161, Art. 314.

To fix the ideas assume that the three vertices X, Y, Z of (Q) are collinear.

Let (XX'), (YY'), (ZZ') denote the circles having XX', YY', ZZ' for diameters. With A as center draw a circle (A) of arbitrary radius; with B and C as centers draw the circles (B), (C) so that the triad of circles (A), (ZZ'), (B) and (A), (YY'), (C) shall be coaxal. The three circles (A), (B), (C) so drawn satisfy the conditions of the problem.

Of the three coaxal circles (A), (ZZ'), (B), the circle (ZZ') divides the line of centers of the other two harmonically, by construction, hence (Art. 1b) the points Z, Z' are the centers of similitude of the circles (A), (B). Similarly, the points Y, Y' are the centers of similitude of the circles (A), (C).

The points Y, Z being centers of similitude of the pairs of circles (A), (C) and (A), (B), the point $X=(YZ, BC)$ is a center of similitude of the circles (B), (C), and the same holds for the harmonic conjugate X' of X for B, C. The proposition is proved.

3. *The circles of similitude of three circles taken two-by-two are coaxal.**

The points B, C (Art. 2a) are inverse with respect to the circle of similitude (XX') of (B), (C), hence any circle passing through B, C, and in particular the circle $(O)=ABC$ is orthogonal to (XX').

The circle (XX') being coaxal with (B), (C), is orthogonal to any circle orthogonal to (B) and (C); in particular (XX') is orthogonal to the orthogonal circle (R) of the three circles (A), (B), (C).

Thus the circles (O) and (R) are both orthogonal to (XX'), and also to (YY') and (ZZ'), for analogous reasons, hence the three circles of similitude (XX'), (YY'), (ZZ') are coaxal.

4. Corollary. Newton's theorem. ***The mid-points of the three segments determined by the three pairs of opposite vertices of a complete quadrilateral are collinear.***

5. Bibliographical note. J. Steiner states this proposition (Art. 4), without proof, in Gergonne's Annales de Mathématiques, Vol. 18, 1828, p. 302. He attributes the proposition to Newton, without indicating any specific reference to Newton's works.

The proposition was given by J. T. Connor in the Ladies' Diary for 1795, according to J. S. Mackay, Proceedings of the Edinburgh Mathematical Society, Vol. 9, 1890-91, footnote on p. 82.

In Gergonne's Annales de Mathématiques Vol. 1, 1810-1811, p. 314 the proposition is proved analytically by Rochat, together with

*CG. p. 197, Art. 394.

some other properties of the complete quadrilateral. A synthetic proof of the proposition is also given there by Vecten on p. 316.

About the same time the same proposition was pointed out by Gauss in Monatliche Korresp., Vol. 22, 1810, p. 115. See R. Baltzer, *Elemente der Mathematik,* Vol. 2, 1883, p. 58, footnote.

The proposition has since been proved in a variety of ways by a great many writers, some of them with names of great distinction, like Poncelet, Chasles, J. J. Sylvester.

The line joining the three mid-points is frequently referred to as the Newton line of the complete quadrilateral.

The line may also be said to be the Newton line of the three circle (A), (B), (C) (Art. 2b.)

6. Converse of Newton's theorem. *If on each side of a given triangle a pair of points are marked harmonically separated by the corresponding vertices and so that the mid-points of the three segments obtained are collinear, the three pairs of points are the vertices of a complete quadrilateral.*

Let the points X and X', Y and Y', Z and Z' be harmonically separated by the points B and C, C and A, A and B: let the mid-points X_0, Y_0, Z_0 of the segments XX', YY', ZZ' be collinear. Construct the circles (XX'), (YY'), (ZZ'), (A), (B), (C) as before (Art. 2b). It follows from this construction that Y and Y', Z and Z' are the centers of similitude of the pairs of circles (A), (C) and (A), (B), respectively.

The three circles (XX'), (YY'), (ZZ') with collinear centers X_0, Y_0, Z_0 are orthogonal to the circle (O) (Art. 3), hence the three circles are coaxal. But of these three circles (YY') and (ZZ') are orthogonal to the circle (R) (Art. 3), hence the same holds for (XX'). Thus the three circles (B), (C), (XX') with collinear centers B, C, X_0 are all orthogonal to (R), hence they are coaxal, therefore the points X, X' are the centers of similitude of the circles (B), (C) (Art. 1b).

Thus the three pairs of points X and X', Y and Y', Z and Z' are the centers of similitude of the three circles (A), (B), (C) taken two-by-two, hence the proposition (Art. 2a).

7. Theorem. *With the vertices of a given triangle as centers three circles may be drawn so that they will have a given line as their Newton line.*

Let X_0, Y_0, Z_0 be the traces of the given line on the sides BC, CA, AB of the given triangle ABC.

With Y_0, Z_0 as centers draw the circles (Y_0), (Z_0) orthogonal to the circle $(O)=ABC$, and with A as center and an arbitrary radius draw a circle (A). This circle and the circles (B), (C) with B, C as

centers and coaxal, respectively, with the pairs of circles (A), (Z_0) and (A), (Y_0) satisfy the required conditions.

Indeed, the circles (Z_0), (O) being orthogonal, the points A, B are inverse with respect to (Z_0) and are therefore harmonically separated by the traces Z, Z' of (Z_0) on the line AB. But (Z_0) is coaxal with (A) and (B), by construction, hence it is the circle of similitude of (A), (B) (Art. 1b). Similarly (Y_0) is the circle of similitude of (A), (C).

Thus Z_0, Y_0 are two points of the Newtonian line of the three circles (A), (B), (C).

8. Corollary. *A complete quadrilateral is determined by its diagonal triangle and its Newton line.*

9. Newton's theorem (Art. 4) may be stated in the following projective form: *The harmonic conjugates, with respect to the pairs of opposite vertices of a complete quadrilateral, of the points of intersection of the sides of its diagonal triangle with a given transversal, are collinear.**

B.

10. We have considered above (Art. 2b) a complete quadrilateral as determined by three pairs of points situated on the three sides of a triangle, separated harmonically by the corresponding vertices, and such that three of the six points are collinear.

Conceived in this manner the notion of the complete quadrilateral may readily be extended to space. Moreover, the properties of the complete quadrilateral proved above apply to the generalized figure, as will be shown in what follows.

Given the tetrahedron $(T) = ABCD$, let X, Y, Z, U, V, W be the traces of a given plane on the edges BC, CA, AB, DA, DB, DC of (T), and X', Y', $\cdots$, W' the harmonic conjugates of $X, Y, \cdots, W$ for the corresponding pairs of vertices of (T). The twelve points $X, Y, \cdots, W'$ form a desmic system; the pairs of points X, X'; Y, Y'; $\cdots$ W, W' may be said to be the pairs of opposite vertices of the system, and (T) its diagonal tetrahedron.†

11. a. *The centers of similitude of four spheres taken in pairs form a desmic system the diagonal tetrahedron of which has for vertices the centers of the given spheres.*‡

*Boletin Matematico (Buenos Aires), Vol. 2, 1929, p. 80, Q. 148; Vol. 14, 1941, p. 283.

†For the theory and bibliography of a desmic system see *MPSG.*, pp. 230-240 and p. 302, Art. 706.

‡Cf. *MPSG.*, pp. 157 ff.

b. Converse Theorem. *With the vertices of the diagonal tetrahedron of a given desmic system of points as centers four spheres may be drawn so that the pairs of opposite vertices of the system will be the centers of similitude of the four spheres taken in pairs.*

The proof is analogous to the case of the plane (Art. 2b).*

12. a. The six spheres of similitude of four given spheres taken in pairs belong to the same coaxal net.

The proof is analogous to the case of the plane (Art. 3).†

b. Corollary. Newton's theorem in space. *The six mid-points of the six segments determined by the pairs of opposite vertices of a desmic system of points, are coplanar.*‡

The plane of the six points may be referred to as the Newtonian plane of the desmic system of points.

The plane may also be said to the Newtonian plane of the four spheres considered.

13. Converse of Newton's Theorem in Space. *If on the edges of a tetrahedron pairs of points are marked harmonic to the respective pairs of vertices and so that the mid-points of the six segments so marked are coplanar, the six pairs of points marked form a desmic system.*

Let X, X'; Y, Y'; Z, Z'; U, U'; V, V'; W, W' be pairs of points harmonically separated by the pairs of vertices B, C; C, A; A, B; D, A; D, B; D, C of the given tetrahedron $(T) = ABCD$, and suppose that the midpoints $X_0, Y_0, \cdots, W_0$ of the segments $XX', YY', \cdots, WW'$ lie in a plane, say, (L).

Construct the spheres (UU'), (VV'), (WW') having the segments UU', VV', WW' for diameters; with D as center draw a sphere (D), of arbitrary radius. With the points A, B, C as centers draw the spheres (A), (B), (C) respectively coaxal with the pairs of spheres $(D), (UU')$; $(D), (VV')$; $(D), (WW')$.

It follows from this construction that the pairs of points U, U'; V, V'; W, W' are the centers of similitude of the pairs of spheres $(D), (A)$; $(D), (B)$; $(D), (C)$ (Art. 1b). Let us now show that the points X, X' are the centers of similitude of the spheres (B), (C).

The points B, C and X, X' are harmonic, by construction, hence the sphere $(O) = ABCD$ is orthogonal to the sphere (XX') having XX' for diameter; the sphere (O) is also orthogonal to (VV'), (WW'), for

*Cf. *MPSG.*, p. 206, Art. 642.
†Cf. *MPSG.*, p. 203, Art. 634.
‡Cf. Mathematics Student (Madras, India), Vol. 3, 1935, p. 98, Art. 4.

analogous reasons. Now the points X_0, V_0, W_0 are collinear, for they lie on the line of intersection of the plane DBC with the plane (L), hence the three spheres (XX'), (VV'), (WW') are coaxal.

Let (R) be the orthogonal sphere of the four spheres (A), (B), (C), (D). The sphere (VV') being coaxal with (D), (B), is orthogonal to (R), and so is (WW') orthogonal to (R), for like reasons. Thus (R) is orthogonal to the first two of the three coaxal spheres (VV'), (WW'), (XX'), hence (R) is orthogonal to (XX').

It follows that the three spheres (XX'), (B), (C) with collinear centers are orthogonal to the same sphere (R), hence they are coaxal, and therefore (XX') is the sphere of similitude of (B), (C) (1b). Thus the points X, X' are the centers of similitude of (B), (C).

In a like manner we may show that the points Y, Y' and Z, Z' are centers of similitude of the pairs of spheres (C), (A) and (A), (B), hence the proposition (Art. 11a).

14. Theorem. *With the vertices of a given tetrahedron as centers four spheres may be drawn so that they will have a given plane as their Newton plane.*

Let X_0, Y_0, Z_0, U_0, V_0, W_0 be the traces of the given plane (L) on the edges BC, CA, AB, DA, DB, DC of a given tetrahedron $(T)=ABCD$. With the points U_0, V_0, W_0 as centers draw the spheres (U_0), (V_0), (W_0) orthogonal to the sphere $(O)=ABCD$. With D as center draw the sphere (D), of arbitrary radius. This sphere and the spheres (A), (B), (C) with A, B, C as centers and respectively coaxal with the pairs of spheres (D), (U_0); (D), (V_0); (D), (W_0) satisfy the required conditions.

Indeed, the spheres (U_0), (O) being orthogonal, the traces U, U' of (U_0) on DA are harmonic to the points D, A, hence (U_0) is the sphere of similitude of (D), (A) (Art. 1b) and U_0 is a point of the Newtonian plane of the four spheres (A), (B), (C), (D).

In a similar manner we may show that the points V_0, W_0 are points on the Newtonian plane of those four spheres. Thus the given plane (L) contains three non-collinear points of the Newtonian plane and therefore is identical with that plane.

The radius of (D) having been taken arbitrarily, there are an infinite number of sets of four spheres satisfying the given conditions, but the radii of the spheres in any two such sets are proportional.

15. Corollary. *A desmic system of points is determined by its diagonal tetrahedron and its Newtonian plane.*

16. Newton's theorem in space (Art. 12b) may be stated in the following projective form: *The six harmonic conjugates, with respect to*

the pairs of opposite vertices of a desmic system of points, of the points of intersection of the edges of the diagonal tetrahedron with a given transversal plane, are coplanar.

17. Application. Given a tetrahedron $(T)=ABCD$ and a sphere (L), there are two spheres tangent to the edge BC and coaxal with the spheres (L) and $(O)=ABCD$. *The two points of contact and the analogous pairs of points on the five other edges of (T) are six pairs of opposite vertices of a desmic system.*

Let X_0 be the trace on BC of the radical plane of the spheres (L), (O). If a sphere (T) is to be coaxal with (L), (O) and touch BC, say, in T, we must have

$$X_0B \cdot X_0C = X_0T^2.$$

We can therefore find two positions for T, say X and X', satisfying this condition; the points X, X' will be equi-distant from X_0 and harmonically separated by B, C. The points X, X' will be real, if X_0 lies outside the segment BC.

Similarly for the other edges of (T). The six pairs of points will thus form a desmic system of which the radical plane of the spheres (L), (O) will be the Newtonian plane. (Art. 13).

18. The remarkable analogies which exist between the complete quadrilateral and the desmic system of points suggest that it may be possible to extend the properties considered above to spaces of higher dimensions.

Conical Roulettes

By R. P. Johnson
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1. *Introduction.* Since the intersection of two quadric surfaces will in general be a quartic, the intersection of two ruled quadrics with a common ruling will factor into a linear curve, the common ruling, and a cubic. If the quadrics have two common rulings, their curve of intersection will consist of the two common rulings and a conic. In particular, if one of the ruled quadrics is a cone, called the principal cone, and if it is rolled on a plane with its vertex at a fixed point, the curve of intersection will trace a curve in the plane, the so-called conical roulette. In the discussion, both quadrics will be taken as cones.

2. *The general case.* Let the equation of the principal cone be

$$ay^2+bz^2+2cxz+2hyz=0$$

and let the equation of the other cone be

$$Ay^2+Bz^2+2Cz(x-g)+2Dyz+2Ey(x-g)=0$$

where both equations represent proper cones. The x axis is the ruling common to both.

The plane $z=\lambda y$ cuts each cone in the x axis and in an additional element and these elements intersect in a point on the cubic whose parametric equations in homogeneous coordinates are

$$\begin{aligned}
\rho x &= g(a+2h\lambda+b\lambda^2)(E+C\lambda)\\
\rho y &= -2gc\lambda(E+C\lambda)\\
\rho z &= -2gc\lambda^2(E+C\lambda)\\
\rho t &= (a+2h\lambda+b\lambda^2)(E+C\lambda)-c\lambda(A+2D\lambda+B\lambda^2)\\
&= aE+(aC-cA+2hE)\lambda+(bE+2hC-2cD)\lambda^2\\
&\quad +(bC-cB)\lambda^3.
\end{aligned}$$

This is a proper cubic since its discriminant does not vanish.

The character of the cubic is determined by the nature of the roots of $t=0$. If we use the notation associated with the solution of a cubic equation, the curve may be classified as follows:

$\Delta > 0$, one root of $t=0$ is real and two are imaginary and the curve is a Cubical Ellipse.

$\Delta < 0$, there are three real and different roots of $t=0$ and the curve is a Cubical Hyperbola.

$\Delta = 0, G, H$ not both zero, the roots of $t=0$ are all real and two are equal and the curve is a Cubical Hyperbolic Parabola.

$\Delta = G = H = 0$, the roots of $t=0$ are all real and equal and the curve is a Cubical Parabola.

The distance R from the vertex of the principal cone to a point on the cubic is given by

$$(1) \qquad R = \frac{g(E+C\lambda)\sqrt{(a+2h\lambda+b\lambda^2)^2+4c^2\lambda^2+4c^2\lambda^4}}{(E+C\lambda)(a+2h\lambda+b\lambda^2)-c\lambda(A+2D\lambda+B\lambda^2)}.$$

If the discriminant of the second cone vanishes, the cone degenerates into the planes

$$(2CD-BE)y+2C^2(x-g)+BCz=0 \qquad \text{and}$$

$$(2) \qquad Ey+Cz=0$$

and since the discriminant of the cubic will also vanish, the curve of intersection will be a plane curve whose equations in homogeneous coordinates are

$$\sigma x = gC^2(a+2h\lambda+b\lambda^2)$$

$$\sigma y = -2gcC^2\lambda$$

$$\sigma z = -2gcC^2\lambda^2$$

$$\sigma t = C^2(a+2h\lambda+b\lambda^2)-c\lambda(2CD-BE+BC\lambda).$$

The curve is a conic and is classified in the usual manner by considering the nature of the roots of $t=0$. The plane (2) cuts the cone in two rulings, one of which is the x axis.

The distance from the vertex of the principal cone to a point on the conic is given by

$$(3) \qquad R = \frac{gC^2\sqrt{(a+2h\lambda+b\lambda^2)^2+4c^2\lambda^2+4c^2\lambda^4}}{C^2(a+2h\lambda+b^2\lambda^2)-c\lambda(2CD-BE+BC\lambda)}.$$

When the principal cone is rolled about its vertex, a point on the curve of intersection will correspond to a point in the plane whose coordinates are (R,ϕ). The value of R is given by (1) or (3) as a function of λ.

In order to express ϕ as a function of λ, consider the curve of intersection of the cone and the plane $x=1$. Its polar equation in the plane is

$$r=\frac{-2c\sin\Theta}{a\cos^2\Theta+b\sin^2\Theta+2h\sin\Theta\cos\Theta}$$

where $\tan\Theta=\lambda$ and the pole is at (1,0,0).

The elimination of R, r and $dr/d\Theta$ from the relations

$$(ds/d\phi)^2=R^2+(dR/d\phi)^2$$

$$(ds/d\Theta)^2=r^2+(dr/d\Theta)^2$$

and
$$R^2=1+r^2$$

gives
$$N_1^2(d\phi/d\Theta)^2=16c^2N_2D_1^2$$

where N_1, N_2, D_1 are functions of Θ. By the substitution

$$\tan 2\Theta=(p+qu)/(1+u) \quad \text{this becomes}$$

$$(d\phi)^2=4c^2(q-p)^2N_3D_3^{-2}(du)^2 \tag{4}$$

where
$$N_3=A_4u^4+A_3u^3+A_2u^2+A_1u+A_0$$

and
$$D_3=b_4u^4+b_3u^3+b_2u^2+b_1u+b_0,\ A_n,\ b_n$$

functions of p and q. Since p and q are arbitrary constants, we can choose them to satisfy $A_3=0$, $A_1=0$. If p_1, q_1 are real solutions of these equations, (4) becomes

$$d\phi=2c(q_1-p_1)D_3^{-1}\sqrt{A_4u^4+A_2u^2+A_0}du \tag{5}$$

$$=2c(q_1-p_1)D_3^{-1}\sqrt{P(u)du}.$$

If $P(u)$ is a perfect square, (5) is integrable in terms of elementary functions. Assuming $P(u)$ is not a perfect square, the right hand side of (5) breaks up into

$$2c(q_1-p_1)[R_1(u^2)/\sqrt{P(u)}-uR_2(u^2)/\sqrt{P(u)}]\ du$$

where R_1 and R_2 are rational functions. The integration of the first term gives elliptic integrals of the first, the second and the third kinds and elementary functions. The second term integrates in terms of elementary functions.

Since ϕ can be expressed as a function of λ, the values of ϕ and R in terms of λ give the parametric equations of the roulette.

3. *One cone specialized.* There is no loss in the generality of either the space cubic or the space conic when the principal cone is specialized. Let the equation of the principal cone be

$$ay^2+bz^2+2czx=0$$

subject to the condition that it be a proper cone and also that

$$c^2+ab-a^2=0$$

in order that it be right circular. Let the equation of the other cone be

$$Ay^2+Bz^2+2Cz(x-g)+2Dyz+2Ey(x-g)=0.$$

The plane $z=\lambda y$ cuts each cone in an element other than the x axis and the intersection of the corresponding elements gives a point on the space cubic whose equations in homogeneous coordinates are

$$\begin{aligned}
\rho x &= g(a+b\lambda^2)(E+C\lambda) \\
\rho y &= -2gc\lambda(E+C\lambda) \\
\rho z &= -2gc\lambda^2(E+C\lambda) \\
\rho t &= (a+b\lambda^2)(E+C\lambda)-c\lambda(A+2D\lambda+B\lambda^2)
\end{aligned} \tag{6}$$

where the nature of the cubic is determined in the usual way.

If the second cone degenerates into two planes, the resulting curve is a conic whose equations in homogeneous coordinates are

$$\begin{aligned}
\sigma x &= gC^2(a+b\lambda^2) \\
\sigma y &= -2gcC^2\lambda \\
\sigma z &= -2gcC^2\lambda^2 \\
\sigma t &= C^2(a+b\lambda^2)-c\lambda(2CD-BE+BC\lambda)
\end{aligned} \tag{7}$$

where the character of the conic is determined as usual.

Since the principal cone is right circular, a plane perpendicular to its axis and passing through $L(R,0,0)$ cuts it in a circle. The plane $z=\lambda y$ cuts the circle in two points P and L and the chord $PL=r$ has for its direction cosines

$$\begin{aligned}
\cos\alpha &= c\lambda/M \\
\cos\beta &= a/M \\
\cos\gamma &= a\lambda/M \quad \text{where} \quad M=\sqrt{a^2+(a^2+c^2)\lambda^2}.
\end{aligned}$$

By means of the relations

$$r=-2R\cos\alpha=-2cR\lambda/M$$

and $$(R\,d\phi)^2=(r\,d\beta)^2+(dr)^2,$$

it follows that $$d\phi=-2ac/M\,d\lambda$$

and, on integration,

$$\phi=-2c(a^2+c^2)^{-\frac{1}{2}}\tan^{-1}(a^{-1}\lambda\sqrt{a^2+c^2}).$$

Let Φ be the angle between two consecutive maps of the same element of the cone. The case of greatest interest is that for which $m\Phi=2\pi$, m a positive integer. Then

$$\phi=2m^{-1}\tan^{-1}(m\lambda/\sqrt{m^2-1}).$$

Equations (6) and (7) in terms of m become

$$\begin{aligned}
\text{(8)}\qquad \rho x&=g[m^2-1+(m^2-2)\lambda^2](E+C\lambda)\\
\rho y&=2g\sqrt{m^2-1}\lambda(E+C\lambda)\\
\rho z&=2g\sqrt{m^2-1}\lambda^2(E+C\lambda)\\
\rho t&=[m^2-1+(m^2-2)\lambda^2](E+C\lambda)\\
&\quad+\sqrt{m^2-1}\lambda(A+2D\lambda+B\lambda^2)
\end{aligned}$$

and

$$\begin{aligned}
\text{(9)}\qquad \sigma x&=gC^2[m^2-1+(m^2-2)\lambda^2]\\
\sigma y&=2gC^2\sqrt{m^2-1}\lambda\\
\sigma z&=2gC^2\sqrt{m^2-1}\lambda^2\\
\sigma t&=C^2[m^2-1+(m^2-2)\lambda^2]\\
&\quad+\sqrt{m^2-1}\lambda(2CD-BE+BC\lambda).
\end{aligned}$$

The parametric polar equations of the roulette of the cubic or the conic are given by

$$\begin{aligned}
\text{(10)}\qquad R&=\sqrt{x^2+y^2+z^2}/t\\
\phi&=2m^{-1}\tan^{-1}(m\lambda/\sqrt{m^2-1})
\end{aligned}$$

where x, y, z and t are given by (8) or (9).

If we eliminate λ between (8) and (10), the polar equation of the roulette of the space cubic may be written

$$RD_4=gm^2[mE+\sqrt{m^2-1}\,C\tan m\phi/2]\sec^2 m\phi/2$$

where
$$\begin{aligned}
D_4&=m^3E+m^2(\sqrt{m^2-1}\,C+A)\tan m\phi/2\\
&\quad+m[(m^2-2)E+2\sqrt{m^2-1}\,D]\tan^2 m\phi/2\\
&\quad+\sqrt{m^2-1}[(m^2-2)C+\sqrt{m^2-1}\,B]\tan^3 m\phi/2.
\end{aligned}$$

The curve passes through the origin tangent to the lines

$$\phi = 2m^{-1}\tan^{-1}(-mE/c\sqrt{m^2-1}).$$

R is infinite when $\qquad \phi = 2m^{-1}\tan^{-1}(m\lambda_1/\sqrt{m^2-1})$

where λ_1 is a root of $t=0$ in (8). There will be an asymptote determined by this value of ϕ if the polar subtangent S_t is finite. The condition for the asymptote is

$$\frac{-2g\sqrt{m^2-1}\,E+C\lambda_1}{\left\{\begin{array}{l}\sqrt{m^2-1}(\sqrt{m^2-1}\,C+A)+2\lambda_1[(m^2-2)E+2\sqrt{m^2-1}\,D]\\ \quad +3\lambda_1^2[(m^2-2)C+\sqrt{m^2-1}\,D]\end{array}\right\}} \neq \infty.$$

By eliminating λ between (9) and (10), the polar equation of the roulette of the space conic may be written

$$RD_5 = gC^2m^2\sec^2 m\phi/2$$

where

$$D_5 = m^2C^2 + m(2CD-BE)\tan m\phi/2$$
$$+C[(m-2)C+\sqrt{m^2-1}\,B]\tan^2 m\phi/2.$$

This curve does not pass through the pole. The condition for an asymptote is

$$\frac{-2gC^2\sqrt{m^2-1}}{\sqrt{m^2-1}(2CD-BE)+2C[(m^2-2)C+\sqrt{m^2-1}\,B]\lambda_1} \neq \infty$$

where λ_1 is a root of $t=0$ in (9).

4. *Numerical examples.* In all the examples, $m=3$ and the principal cone has for its equation $8y^2+7z^2-4\sqrt{2}zx=0$. (a) The cubical ellipse (Fig. 1). Let the equations of the cones be

$$8y^2+7z^2-4\sqrt{2}zx=0$$

and

$$\sqrt{2}y^2+\sqrt{2}z^2+y(x-5)-z(x-5)=0.$$

The equations of the cubic are

$$\rho x = -35\lambda^3+35\lambda^2-40\lambda+40$$

$$\rho y = -20\sqrt{2}\lambda^2+20\sqrt{2}\lambda$$

$$\rho z = -20\sqrt{2}\lambda^3+20\sqrt{2}\lambda^2$$

$$\rho t = \lambda^3+7\lambda^2+8.$$

Since $\Delta > 0$, the curve is a cubical ellipse. The equation of the roulette is

$$R = \frac{45(3-2\sqrt{2}\tan 3\phi/2)\sec^2 3\phi/2}{27+21\tan^2 3\phi/2+2\sqrt{2}\tan^3 3\phi/2}\,.$$

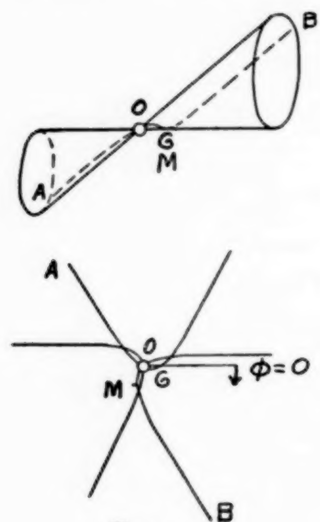

Fig 1

Since the equation $t=0$ has only one real root, $\lambda = -7.15$ approximately and the subtangent is finite, the part of the roulette, corresponding to a complete revolution of the cone, will have an asymptote corresponding to $\phi = 61+°$. One revolution of the cone gives the curve $GOABM$.

(b) The cubical hyperbola (Fig. 2). Let the equations of the cones be

$$8y^2+7z^2-4\sqrt{2}zx=0$$

and $\quad 332y^2+224z^2-511yz+84\sqrt{2}y(x+10)-96\sqrt{2}z(x+10)=0.$

The equations of the cubic are

$$\rho x = -840\lambda^3+735\lambda^2-960\lambda+840$$

$$\rho y = -480\sqrt{2}\lambda^2+420\sqrt{2}\lambda$$

$$\rho z = -480\sqrt{2}\lambda^3+420\sqrt{2}\lambda^2$$

$$\rho t = -14(\lambda-1)(\lambda-6)(2\lambda+1).$$

Since $\Delta < 0$, the curve is a cubical hyperbola. The equation of the roulette is

$$R = -\frac{270}{7}\,\frac{(21-16\sqrt{2}\tan 3\phi/2)\sec^2 3\phi/2}{16\sqrt{2}\tan^3 3\phi/2-156\tan^2 3\phi/2+45\sqrt{2}\tan 3\phi/2+81}$$

Since $t=0$ has three real and distinct roots and since the corresponding subtangents are finite, the part of the roulette corresponding to a complete revolution of the cone, will have asymptotes corresponding to $\phi=31+°$, $54+°$ and $101+°$. One complete revolution of the cone gives the curve $OABCDEFO$.

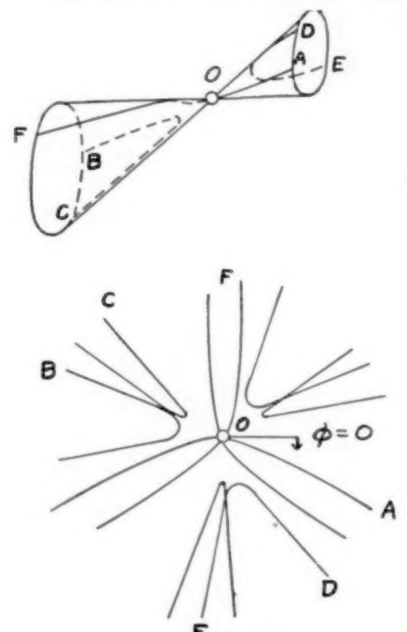

Fig 2

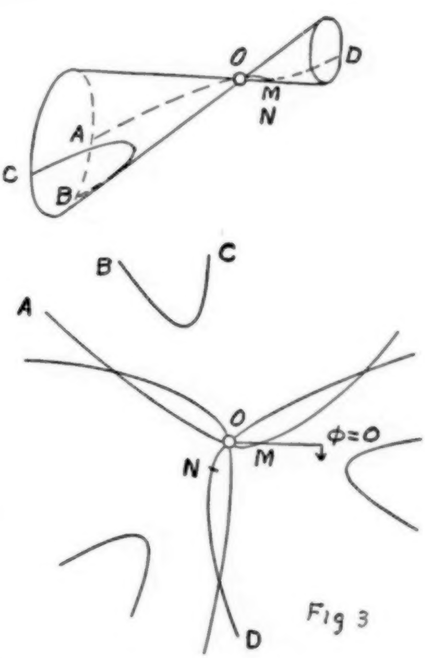

Fig 3

(c) The cubical hyperbola parabola (Fig. 3). Let the equations of the cones be

$$8y^2+7z^2-4\sqrt{2}zx=0$$

and $$16y^2+16z^2-13yz+4\sqrt{2}y(x-10)-8\sqrt{2}z(x-10)=0.$$

The equations of the cubic are

$$\rho x=-70\lambda^3+35\lambda^2-80\lambda+40$$
$$\rho y=-40\sqrt{2}\lambda^2+20\sqrt{2}\lambda$$
$$\rho z=-40\sqrt{2}\lambda^3+20\sqrt{2}\lambda^2$$
$$\rho t=(\lambda+1)(\lambda-2)^2.$$

Since $\Delta = 0$, $G, H \neq 0$, the curve is a cubical hyperbolic parabola. The equation of the roulette is

$$R = \frac{90(3\sqrt{2} - 8 \tan 3\phi/2) \sec^2 3\phi/2}{27\sqrt{2} - 18\sqrt{2} \tan^2 3\phi/2 + 8 \tan^3 3\phi/2}.$$

The equation $t = 0$ has a single root, $\lambda = -1$, and a double root, $\lambda = 2$. The subtangent, corresponding to the first root is finite, so the part of the roulette, given by one revolution of the cone, will have an asymptote corresponding to $\phi = 89+°$. Since the second root gives an infinite subtangent, there is no corresponding asymptote. One complete revolution of the cone gives the curve $MOABCDN$.

(d) The cubical parabola (Fig. 4). Let the equations of the cones be

$$8y^2 + 7z^2 - 4\sqrt{2}zx = 0$$

and

$$20y^2 + 8z^2 + yz - 4\sqrt{2}y(x - 10) - 4\sqrt{2}z(x - 10) = 0.$$

The equations of the cubic are

$$\begin{aligned} \rho x &= 70\sqrt{2}\lambda^3 + 70\sqrt{2}\lambda^2 + 80\sqrt{2}\lambda + 80\sqrt{2} \\ \rho y &= 40\sqrt{2}\lambda^2 + 40\sqrt{2}\lambda \\ \rho z &= 40\sqrt{2}\lambda^3 + 40\sqrt{2}\lambda^2 \\ \rho t &= -(\lambda - 2)^3. \end{aligned}$$

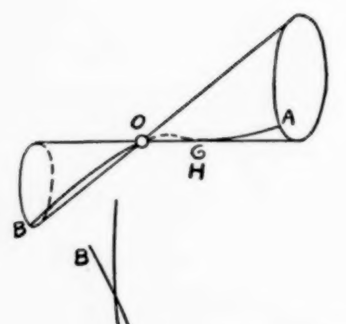

Fig 4

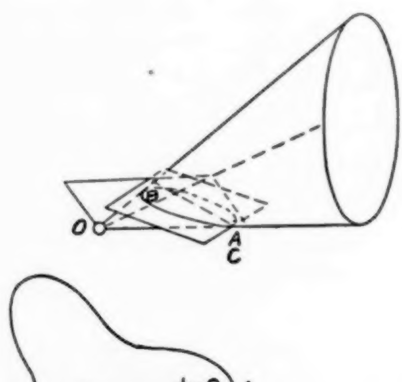

Fig 5

Since $G=H=\Delta=0$, the curve is a cubical parabola. The equation of the roulette is

$$R=\frac{90(3\sqrt{2}+4\tan 3\phi/2)\sec^2 3\phi/2}{27\sqrt{2}-54\tan 3\phi/2+18\sqrt{2}\tan^2 3\phi/2-4\tan^3 3\phi/2}.$$

Since the equation $t=0$ has a triple root and since the corresponding subtangent is infinite, there is no asymptote in the finite part of the plane. One complete revolution of the cone gives the curve *GABOH*.

(e) The ellipse (Fig. 5). Let the equations of the cones be

$$8y^2+7z^2-4\sqrt{2}zx=0$$

and
$$5z^2+10yz+2z(x-5)+4y(x-5)=0$$

or
$$(2x+5z-10)(2y+z)=0.$$

The equations of the curve are

$$\sigma x=35\lambda^2+40$$
$$\sigma y=20\sqrt{2}\lambda$$
$$\sigma z=20\sqrt{2}\lambda^2$$
$$\sigma t=(10\sqrt{2}+7)\lambda^2+8.$$

Since the discriminant of $t=0$ is negative, the curve is an ellipse. The equation of the roulette is

$$R=\frac{45\sec^2 3\phi/2}{9+(10\sqrt{2}+7)\tan^2 3\phi/2}.$$

The roulette has no asymptotes since $t=0$ has no real root. One complete revolution of the cone gives the curve *ABC*.

A Fifth Lesson in the History of Mathematics

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In the third of these lessons, volume 15, page 234 (1941), we considered various forward steps in the long history of the solution of the quadratic equation extending through a period of about four thousand years. In the present lesson we shall consider similarly various steps in the history of the solution of the cubic equation, which extend almost as far back as those of the quadratic equation, but are quite different in many respects as might be expected from the fact that they involve more difficult mathematical questions. Concepts which have influenced the development of mathematics for such long periods of time may well be regarded as central in the history of our subject. Reasons for their very slow development are closely related to the study of the intellectual development of the human race and form an important part of such a study. The full richness of the history of mathematics involves a consideration of earlier backwardness implied by its advances.

The solution of the general cubic equation as well as that of the general quadratic equation could obviously not have been completed either algebraically or geometrically before the theory of our ordinary complex numbers was understood, and these solutions were very strong factors tending to enrich the mathematical world with the extension of the number concept. It is now commonly assumed that the theory of our ordinary complex numbers was not fully understood before about the beginning of the nineteenth century and this assumption is a central fact in the history of the efforts of the earlier peoples to solve the quadratic and the cubic equations. It throws light not only on such efforts on the part of the ancient Greeks but also on similar efforts on the part of well known later mathematicians, such as

René Descartes (1596-1650), Isaac Newton (1642-1727), and Leonhard Euler (1707-1783), which are greatly illuminated thereby.

The efforts to secure an insight into the nature of our ordinary complex numbers seem to have been more influenced by the attempts to solve the cubic equation than by those to solve the quadratic equation notwithstanding the fact that the latter solution is the simpler and is commonly met by the modern student of our subject earlier than the general solution of the cubic equation. This is an instance where the actual historical facts are not in agreement with the natural preconceptions of the student of mathematics who has not acquainted himself with the history of its development. The correction of natural preconceptions constitutes one of the most interesting elements of the study of the history of mathematics, but this history is also rich in the support of what might reasonably have been expected. It is therefore desirable to emphasize variations therefrom as well as agreements therewith.

Although the general solution of the quadratic and the cubic equations could not be fully understood before the theory of the ordinary complex numbers became known it should be emphasized that many such special equations were solved correctly long before this time. Since a cubic equation with real coefficients has always at least one real root while a quadratic equation with real coefficients does not necessarily have a real root it is clear that in early times, when generally only one root of an equation was sought, the quadratic equation presented difficulties which did not appear in the study of the cubic equation. When an unknown quantity is raised to powers it may be assumed to associate with itself other unknown quantities which when raised to the same powers are equal thereto as is illustrated by the different roots of unity. Such facts naturally aroused early interest in the study of equations whose degrees exceed unity. The equation has been called a kind of a trap for catching some mathematical unknowns* and this suggests some instructive points of view which are worth considering.

The ancient Babylonians constructed tables of numbers of the form $m^3+n^2=p$. These seem to have served to find one root of each of certain cubic equations. Although the general cubic equation can be reduced to this form by means of the solution of linear and of quadratic equations it is clear that the ancient Babylonians could not have arrived at the general theory of the solution of cubic equations in this way since they could not even solve the general quadratic equations nor could they solve completely this reduced form of the cubic. While

*G. A. Miller, *Collected Works*, Vol. 2., p. 573 (1938).

the ancient Babylonians used general methods in the solution of the quadratic equation but failed to explain these solutions, in general they did not even arrive at the use of general methods in the solution of the general cubic equation. The Italians, however, found such general methods in the first half of the sixteenth century but they could not see the full meaning of the results thus obtained and they have often been given too much credit along this line.

A. C. Clairaut (1713-1765) and J. d'Alembert (1717-1783) were the first to show that Ferro's formula (sometimes incorrectly called Cardan's formula) gives actually the three roots of the general cubic equation and not only one of them as had been commonly assumed earlier. It should be emphasized that both in the case of the solution of the quadratic equation and also in the solution of the cubic equation general methods were used long before they were fully understood and that the results thus obtained naturally gave rise to further advances. The hand has often been ahead of the head in the developments of mathematics in the sense that calculations often gave rise to results which had not been foreseen but whose explanation led to greater theoretic insight.

In the case of the quadratic equation it took more than three thousand years to give a satisfactory explanation of some results obtained by a general method, and the name of the one who first used this method is entirely forgotten. In the case of the cubic equation, on the contrary, the interval between the time when a general method for the solution was first used and the time when the results thus obtained were clearly explained was much shorter, and Scipione del Ferro (1465-1526) is commonly credited with the discovery of this method. In the former case no claims of priority are now known to have ever existed while in the latter case there was a bitter controversy for a considerable time in regard to such claims. This controversy has received considerable attention in some of the histories of mathematics but it is of secondary importance in this history and has often been given too much space therein.

It is well established that a general method for solving algebraically the cubic equation first became widely known as a result of its appearance in the very influential work of H. Cardan (1501-1576) entitled *Ars Magna* (1545). Contrary to what has often been asserted, H. Cardan mentioned the name of N.Tartaglia in this connection and he stated that Scipio del Ferro of Bologna discovered the solution about thirty years earlier. In the same work H. Cardan published a general method for solving algebraically the biquadratic equation, which he credited to his pupil L. Ferrari (1522-1565). As

the *Ars Magna* was printed in Nuremberg in 1544 it seems remarkable that L. Ferrari discovered this method before he was twenty-four years old. This method was based on the solution of the cubic equation and represents the upper limit of such general solutions of equations by methods known before the nineteenth century.

A considerable number of cubic and biquadratic equations appear in the literature of the ancient Babylonians and exhibit their interest in elementary algebra. Their tables of cubes of numbers were necessarily at the same time tables of cube roots and may have served to determine roots of special numerical equations just as did the tables of the form $m^3+n^2=p$ to which we referred above. Their partial solutions of cubic and biquadratic equations were however highly tentative and far from the modern general methods even in form. The ancient Greeks and the Arabs came much closer to an actual solution by their methods based on conic sections. In fact their extensive knowledge of conic sections may have been partly inspired by the fact that this knowledge was useful in finding a real root of certain cubic equations.

Only one cubic equation appears in the extant parts of the noted *Arithmetica* of Diophantus. In modern notation it is

$$x^3-3x^2+3x-1=x^2+2x+3.$$

This can easily be reduced to the form $x(x^2+1)=4(x^2+1)$, from which it follows directly that 4 is the real root while the other two roots are imaginary. These two roots were naturally not considered by Diophantus since the ancient Greeks did not reach the advanced stage of considering complex roots of an equation. In fact, they did not even consider negative roots in any of their extant works. The Arabs advanced beyond the ancient Greeks in their work of finding one positive root of certain cubic equations, but their work along this line was largely influenced by that of Archimedes on the sections of a sphere. Both the Arabians and the Hindus were largely influenced in their mathematical work by earlier work of the Greeks, but they made additions thereto.

An interesting historical fact connected with the solution of the cubic equation is that partial solutions were involved in a public mathematical challenge on the part of Tartaglia and Fior in 1535. These challenges then attracted considerable attention and helped to popularize certain mathematical questions. It is said that the Ambassador from Netherlands once told the king of France that a Belgian mathematician, Adrianus Romanus, propounded a problem relating to the solution of an equation of the 45th degree which no one in

France was able to solve. The king then called F. Vieta (1540-1603), who found 25 of its roots instead of only one. Vieta did not give negative roots. At that time it was not known that an algebraic equation contains as many roots as its degree. This fact was announced, but not proved, in 1608 by Peter Roth. Two of the three great Greek problems of antiquity, viz., the trisection of a general angle, the duplication of the cube and the quadrature of the circle, involve a partial solution of the cubic equation. That is, the first two of these problems involved the finding of a positive root of such an equation while the last is of a transcendental nature. Among the other ancient Greek problems which depend on a partial solution of the cubic equation is the construction of a regular polygon having seven sides. This problem was considered by Archimedes who also considered another cubic problem in the form of the division of a sphere so that the volumes of the parts are in a given ratio, as was noted above. It thus appears again that some connection, between algebra and geometry was noted long before the development of analytic geometry in the seventeenth century, but the Greeks failed to see that there is a wide difference between cubic and quadratic problems as regards constructions with ruler and compasses.

13. *Complex numbers.* By Ferro's rule for finding one root of the cubic equation it is expressed in the form of the sum of two cube roots. When the equation is in the form $x^3+ax=b$ this sum, in modern notation, may be written as follows:

$$\sqrt[3]{\frac{b}{2}+\sqrt{\left(\frac{b}{2}\right)^2-\left(\frac{a}{3}\right)^3}}+\sqrt[3]{\frac{b}{2}-\sqrt{\left(\frac{b}{2}\right)^2-\left(\frac{a}{3}\right)^3}}$$

When the three roots of this equation are real the quantities under the cube root signs are complex and it therefore led to two complex numbers whose sum is a real number. This aroused much interest in calculating with complex numbers. Such calculations are indicated in H. Cardan's *Ars Magna* (1545) but they appear therein with respect to quadratic equations, and it now seems probable that they were based on a work of R. Bombelli which was published in 1572 under the title *l'algebra,* but was in preparation much earlier. It may have been seen in its early stages by H. Cardan while on a visit to Bologna in 1543. At any rate, in his later writings H. Cardan exhibited such a lack of insight into the nature of complex numbers as to give rise to the belief that what he published in his *Ars Magna* thereon was not original with him. His reputation as regards complex numbers was distinguished by these later writings.

In R. Bombelli's algebra there appear rules of calculating with complex numbers which, in modern notation, are as follows:

$$(+)\cdot(+i)=(+i) \qquad (+i)\cdot(+i)=(-)$$
$$(-)\cdot(+i)=(-i) \qquad (+i)\cdot(-i)=(+)$$
$$(+)\cdot(-i)=(-i) \qquad (-i)\cdot(+i)=(+)$$
$$(-)\cdot(-i)=(+i) \qquad (-i)\cdot(-i)=(-)$$

R. Bombelli also solved here many problems relating to the extraction of the cube roots of complex numbers by means of these eight rules and he thus did much towards introducing the formal calculation with these numbers, but he failed to explain these rules. He was professor of mathematics at Bologna but very little is otherwise known in regard to his life and he seems to have been mainly interested in calculations. A second edition of his algebra appeared in 1579 which seems to imply that it was favorably received by the public. It is somewhat singular that he failed to mention therein the name of Ferro in speaking of the partial solution of the cubic equation. In the dispute between Tartaglia and Cardan he properly supported Cardan and his pupil Ferrari.

It took more than three hundred years more for mathematicians to secure a clear insight into the nature of complex numbers notwithstanding the lasting influence along this line which was exerted by the works of H. Cardan and R. Bombelli, and the fact that many mathematicians obtained numerous very interesting formal results in the meantime by the use of these numbers. In particular, in the Latin edition of his noted algebra, J. Wallis gave correctly the three roots of each of the equations $x^3 = \pm 8$, and in 1702 G. W. Leibniz gave correctly the four linear factors of x^4+a^4. In his *Inventions nouvelle* (1629) A. Girard gave somewhat earlier the four zeros of $x^4=4x-3$ and observed that

$$(-1+\sqrt{-2})(-1-\sqrt{-2})=3.$$

The discovery by J. Bernoulli and G. W. Leibniz of the integration of rational fractions by means of the decomposition of their denominators into linear factors attracted the attention of mathematicians to the logarithms of complex numbers. L. Euler knew already in 1728 the interesting formula

$$i^i=e^{-\pi/2}$$

and in 1740 he calculated with real numbers having complex exponents. These calculations were only formal.

For the sake of emphasis we add here that on page 92 of *What is Mathematics?* (1941) by Richard Courant and Herbert Robbins it is stated that "as early as the fifteenth century mathematicians were compelled to introduce symbols for the square roots of negative numbers in order to solve the quadratic and the cubic equations. But they were at a loss to explain the exact meaning of these symbols which they regarded with almost superstitious awe. The name 'imaginary' is a reminder of the fact that these symbols were considered to be somehow fictitious and unreal." There is now, however, no evidence of the use of such symbols in the fifteenth century. In the calculations with the square roots of negative numbers even in the following century by H. Cardan and R. Bombelli it cannot definitely be said that they introduced symbols for the square roots of negative numbers since H. Cardan used the term *per radicem m* for such expressions as $5+\sqrt{-15}$ and $5-\sqrt{-15}$, and R. Bombelli used *piu uia piu di meno, fa piu di meno* for $(+)\cdot(+i)=(+i)$. That is, they used rhetorical algebra in this connection instead of symbolic algebra.

In the theory of logarithms the extensions due to the use of complex numbers were especially striking. While in the domain of real numbers only positive numbers have logarithms and each positive number has only one logarithm, it was noted by L. Euler that every number has an infinite number of complex logarithms. The theory of real logarithms thus became a very special case in the general theory of logarithms. During the eighteenth century there was a great deal of discussion in regard to the logarithms of negative numbers and a considerable amount of controversial literature then appeared on this subject even by leading mathematicians of that time as was noted above. When the questions involved were studied from a more advanced point of view the apparent contradictions disappeared and general agreement was restored. The logarithms of complex numbers naturally arose in the theory of integration by admitting implicitly the rules of integration which had been proved for real numbers.

It was thus that up to about the beginning of the nineteenth century people calculated very successfully with complex numbers with increasing confidence but without establishing the legality of these calculations. J. Wallis made some effort in his *Algebra* (1685) to give a geometric construction of complex numbers but he failed to develop the subject sufficiently to attract general attention and hence his efforts along this line were ineffective. Even the much more extensive developments by C. Wessel (1745)-1818) were for a long time practically unnoticed notwithstanding their usefulness and their rigorous treatment. Somewhat later (1800) J. R. Argand considered the same subject independently in a work which also at first received

little attention even in France. The mathematical world in general did not then seem to realize the great need of a rigorous proof of the correctness of the method, which had been employed in calculating with complex numbers to obtain a large number of very interesting results.

The first proofs of the justification of the use of complex numbers were based on geometric considerations, but in 1837 W. R. Hamilton, (1805-1865) published a theory in the *Transactions of the Irish Academy* which is based on number couples and which has since then been widely adopted. It was read at a meeting of this academy in 1833 and hence this date is commonly given as the date of this theory by means of which complex numbers were satisfactorily introduced from an arithmetic point of view. This was almost 300 years after H. Cardan published the earliest operations with these numbers as roots of a quadratic equation. Among the many others who contributed towards the establishment of complex numbers as a useful mathematical subject are C. F. Gauss in Germany and A. L. Cauchy in France. The former made important applications of these numbers in algebra and in the theory of numbers but he never published their justification as he had promised. The latter published their justification by his theory of algebraic equivalences and thus he gave a new proof of the legitimacy of these numbers.

The influence of Gauss and Cauchy in the first half of the nineteenth century and the many extensions which the use of complex numbers made possible naturally moved these numbers into the forefront of mathematical interest during the nineteenth century as is seen in various developments in number theory and in the functions of a complex variable. Western Europe thus became inspired by ideas which were unknown to the ancients and which were partly based on a multiple number concept instead of that a real positive number to which the ancients had limited their attention. A host of names are favorably connected with the history of the development of complex numbers but the following seven cannot be omitted from any adequate presentation of this history: H. Cardan, R. Bombelli, J. Wallis, C. Wessel, C. F. Gauss, A. L. Cauchy, W. R. Hamilton. Most of these made also other important contributions towards the advancement of mathematics.

One of the most illuminating elements in the study of the history of mathematics at a certain period is the consideration of the development of the number concept at that period, not only in regard to those who at that time had secured the most advanced views of the subject but also in regard to the then common knowledge along this line. In America, for instance, we find that in our earliest mathematical periodi-

cal, the *Mathematical Correspondent*, (1804), it is stated on page 75 of volume 1 that a single quantity cannot be worked with either the positive or the negative sign, and that such a statement as the square of -5 is 25 either signifies no more than that the square of 5 is 25 or it is mere nonsense. Similar views were expressed in some European publications about the same time. A common understanding of the extensions of the number concept became a world inheritance by slow and uncertain steps notwithstanding its fundamenal importance.

The now common symbol i for the positive square root of -1 was used by L. Euler as early as 1777 and, was employed by C. Gauss in his noted *Disquisitiones arithmeticae* 1801. It has since then been commonly adopted in different countries, being used by A. L. Cauchy in France in 1847, who introduced in 1821 the now common term conjugate numbers for two complex numbers of the form $a+bi$ and $a-bi$. J. R. Argand seems to have been the first to use the term modulus for $\sqrt{a^2+b^2}$ when the complex number is written either in the form $a+bi$ or in the form $a-bi$. Attention has frequently been called to the fact that the ancient Greek writer Heron, referred to the $\sqrt{81-144}$ but seemed to regard it as equivalent to $\sqrt{63}$. At any rate this example seems to have had no influence on the later development of complex numbers and may have been due to an error on the part of Heron. A long article on complex numbers appears in *Encyclopedie der Sciences Mathematiques*, tome 1, volume 1, pages 329-468, in which a large number of references may be found. It should be emphasized that for about 250 years the theory of complex numbers was developed by means of the consideration of special cases and when general theories relating thereto were developed the subject had been so well established as to make the general theories appear almost superfluous for a time.

In the combination of our ordinary complex numbers the commutative, associative and distributive laws are satisfied but in such elementary geometric operations as the transformation of an equilateral triangle into itself the commutative law in the combination of these transformations is clearly not always satisfied. This naturally raised the question whether a number system which is more general than our ordinary complex number system would not be useful in mathematical applications. The first number system in which the commutative law in the multiplication of the elements is not preserved but in which the associative law is preserved is the famous system known as quaternions, discovered by the Irish mathematician W. R. Hamilton in 1843. Shortly thereafter H. Grassmann published a system known as *Ausdehnungslehre* which also received considerable attention in various countries. Since then a number of other extensions have been studied by a number of different mathematicians.

The Teachers' Department

Edited by
JOSEPH SEIDLIN, JAMES MCGIFFERT, J. S. GEORGES
and L. J. ADAMS

Trends in Education

BY P. G. ROBINSON
Iowa State College

Our primary schools, not so long ago, were concerned almost exclusively with the teaching of the three R's and our high schools and colleges with the teaching of classical and cultural subjects. There has been a great change. In the secondary field more and more the emphasis has been placed on letting the pupil spend his time in studying that which most interests him. The head of the Education Department of one of our large mid-western universities has said that if the traditional subject matter does not meet with the approval of the pupils, it should be modified or eliminated. The statement that "we teach children instead of subject matter" has come to mean in many schools that the pupils are really getting largely what might be called a "lollypop" education.

I do not believe that we can, nor do we need to try to, justify the teaching of mathematics simply for its discipline. Psychological research would not uphold us in this.

There is a happy medium between the two extremes of the old hickory stick days and the so frequently frothy education of today. I believe, in mathematics for example, that the meat is desirable but that it often has not been, but can be, presented in a manner in which it will be both digestible and pleasant to the palate. More emphasis should be placed on thinking and less on dexterity in manipulation. In my opinion many educators give only lip service to the teaching of thinking. Operations are taught but the student cannot apply them to practical situations. Students of algebra dislike so-called word problems because to solve them they must think instead of using some formula more or less blindly. Examinations, as they are usually given, place a premium on memory and dexterity of manipulation but in the vast majority of cases do not emphasize thinking.

There has been for years a definite trend, especially in our secondary schools, toward teaching less and less mathematics. Only a few years ago the head of our local schools asked permission to allow high school students to graduate without having studied in high school any mathematics at all, in those cases where he thought it desirable.

The Educational Policies Commission of the N. E. A. in a "Report on the Purposes of Education in American Democracy" published in 1938 said, in part, "And what are the children in this school in this age, in this culture, learning? They are learning that the square of the sum of two numbers equals the sum of their squares plus twice their product, etc. For the great majority of boys and girls who are now attending American schools such learning is transitory and of extremely little value".

I asked my class in calculus to compare the capacity of a slat corn crib having a circumference of 50 feet with one having a circumference of 75 feet. Only a few were able to do it without the use of pencil and paper.

Professor E. R. Hedrick of the University of California at Los Angeles says in an article entitled "The Contribution of Mathematics to General Education" in the Mathematics Teacher of January, 1940, "Every tall can in every grocery store is a monument to the women who 'do not need to know any algebra'." Problems involving squares—then cubes—then compound interest offer a fertile and effective field for motivating pupils in their mathematical knowledge of the development of a binomial expansion. As ability increases, annuity problems involving the payment of a fixed sum each year will lead naturally to geometric progressions. Professor Hedrick in the article I previously mentioned says, "I have long believed and maintained that teachers of mathematics must search out and must present to students a variety of situations in which a given mathematical item is applicable. If this is not done, the difficulties of transfer of training will prevent the student from making any great use of the item. To my mind, the failure of teachers to do this is one of the major shortcomings of the whole field of mathematical teaching." Also, another quotation from the same source, "Some educators and curriculum builders have themselves not transferred their own knowledge of these mathematical items sufficiently to realize that they do form the whole basis for public comprehension of these important public issues."

Wherein specfically can we improve our teaching at the algebra level?

One way is to bridge more closely the gap between arithmetic and algebra. The pupil should see that algebra is a method for simplifying

his arithmetic. He should see that algebra is a language, 'the highest development so far achieved in an attempt to express relations with clarity and economy'. I would spend much time in having the beginning (as well as the more advanced) student translate statements into mathematical language. Until the student has learned the fundamental language of mathematics the statements, situations, and diagnostic procedures can, and should be, taken largely from problems vital to all students, such as social problems, consumer problems, taxation, etc. I should not use long lists of drill problems. The dexterity will come when the pupil sees sense and reason in the problems. Furthermore if school administrators and faculty members of our departments of education knew more of mathematics themselves, there probably would be a more wholesome respect for the subject.

In our colleges the subject of specialized vs. generalized knowledge is one about which we are hearing a great deal these days. For years there has been a tendency toward greater and earlier specialization. With the advent of war that tendency has been greatly accelerated. Students in technological schools spend only a minimum of time in getting a general broad comprehensive education and much time in their special fields. A recent editorial in a daily paper says, "Each year hundreds of students graduate from the University of without having actually received an education. They have been given plentiful instruction in their specialized fields, but the general cultural advantages to be gained in the University have been completely ignored."

President Hutchins of the University of Chicago is an outstanding advocate of generalized education. In a recent address entitled "The University and the War" he says, "The chaos in education with which we are familiar is an infallible sign of the disintegration of civilization, for it shows that ideals are no longer commonly held, clearly understood or definitely pursued. To formulate, to clarify, to vitalize the ideals which should animate mankind, this task, which I described not long since as candid and intrepid thinking about fundamental issues, this is the incredibly heavy burden which rests, even in total war, upon the universities. If they cannot carry it, nobody else will, for nobody else can. If it cannot be carried, civilization cannot be saved." In his recent report on 'The State of the University' he says, "It is becoming increasingly clear that the crisis of our time is primarily moral and intellectual. We suffer less from a want of science and technology than from lack of understanding of the aims of life and of organized society. The humanities and the social sciences are those divisions of the University that are capable of dealing with the perplexing moral and intellectual problems that confront the United

States. It may well prove that in the end the fate of our country and the ideals for which it stands will be decided not in the laboratory and the factory, but by candid and intrepid thinking on the issues that lie at the basis of our civilization."

As I see it, one great danger in a too greatly generalized education is in its tendency toward vagueness. One has only to listen to some of our economists to realize this. One way to guard against vagueness is to try to put your thoughts in the form of an equation. Plato says, "If arithmetic, mensuration and weighing be taken away from any art, that which remains will not be much." Lord Kelvin has said, "When you can measure what you are thinking about and express it in numbers, you know something about it. But when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind. It may be the beginning of knowledge but you have scarcely in your thoughts advanced to the stage of science."

But our technological schools are open to criticism. One criticism that is often heard, and I think it is valid, is that the student is so busy with the learning of details and acquiring a technique that he does not have time to think. President Hancher of the University of Iowa said in his inaugural address, "Somewhere the student must be given time to think, to understand, to accomplish. Perhaps he must be freed from certain routines. Perhaps he must know less in order to understand more, although the necessity of such a choice seems doubtful. Perhaps he should have less guidance in order to acquire more mastery. Perhaps less time should be spent in mastering the ideas of others and more in arranging his own. . . ."

Certainly the nation needs the technical expert today and certainly it needs also the man of vision for tomorrow.

It seems to me that the function of an institution of higher learning is twofold. *First:* To give a student at least a speaking acquaintance with the arts, sciences and humanities; to foster a desire in him to know the truth; to teach him to think, weigh and judge; and to create an attitude toward life that will cause him to continue to study and learn after leaving college. *Second:* To give him sufficient specialized training in at least one field so that he may be proficient in that field and even, with more advanced study, a master in that field.

Teaching the Fundamentals of Mathematics

By M. O. Tripp
Wittenberg College

The difficulties in all our mathematical instruction, both in high school and in college, revolve about the matter of prerequisites. Whenever one subject is not understood, as for example first year algebra, there is trouble for the student as long as he continues his mathematical studies. Other branches, in general, do not present the close concatenation of subjects that mathematics does.

In this country a large part of the recitational activities in the mathematical classroom is carried out in connection with blackboard work. In Europe, as a rule, blackboards have not been placed around the room as is common in this country. There the general plan is to have a small blackboard where the master can work in explaining, or one student at a time can work and explain as he proceeds to develop a solution or prove a theorem. It is assumed that the class can learn from listening. The plan of "chalk and talk" is followed in a way quite commonly absent in this country. In this way fundamental principles are continually stressed, while the student is being constantly quizzed by the master on the theorems and processes involved. American teachers send large numbers to the blackboard, frequently all that can work conveniently, and keep them there a large part of the recitation period. The defect in this plan is that the students spend a considerable part of their time juggling symbols. Little opportunity is allowed for a discussion of the ideas underlying these symbols. After a fair amount of work is on the blackboard time should be allowed for class discussion of fundamental ideas. A large part of this discussion should be carried on by students, that is, they should actually learn to *explain* and not merely *read* the equations written on the board.

Computational work in algebra and geometry is one of the topics quite commonly passed over rather superficially; since, to the student, this seems somewhat like a task. The roots of a quadratic equation, when they come out irrational numbers, should first be put in the simplest form and then evaluated to one or two places of decimals.

In plane geometry there is an opportunity to study radicals in a concrete way by finding the sides, apothems, perimeters, and areas

of regular polygons, inscribed and circumscribed about a circle, having a radius given as a numerical value. This kind of work offers training in reading proportions from similar triangles—something in which high school students are weak. It is in this computational work that the student really sees a reason for expressing a fraction with a radical in the denominator as an equivalent fraction with a rational denominator. The argument here is that students do not get fundamentals fixed thoroughly in mind unless they continually resort to numerical approximations.

Notwithstanding arithmetic is quite commonly taught as a mechanical process, all of the instruction should not be of this type. There should be frequent occasions for stressing the fundamental principles. For example, how is a fraction changed in value if we add the same number to both numerator and denominator? By following an inductive study of the problem, students will naturally come to the conclusion that a fraction less than unity is increased by this operation, and a fraction greater than unity is decreased. Then deductive reasoning should be used to prove the theorem true in all cases. This can be carried out somewhat as follows for those who have a little algebra.

For a fraction greater than unity, assume

$$\frac{x}{y}=1+k, \quad k>0. \tag{1}$$

Clear of fractions, $x=y+yk.$

Add c to each side, $x+c=y+c+yk, \quad c>0.$

Divide by $y+c$,

$$\frac{x+c}{y+c}=1+\left(\frac{y}{y+c}\right)k, \tag{2}$$

Evidently the right side of equation (1) is greater than the right side of (2). Hence we have the result:

$$\frac{x}{y}>\frac{x+c}{y+c}.$$

For the case of a proper fraction, in (1) take k negative and greater than -1.

There are certain principles in the teaching of decimals which should be emphasized, in order to give the student a more complete understanding of what he may expect. A rational fraction, in its lowest terms, may always be changed into a terminating decimal provided the denominator contains only the prime factors 2 and 5.

If the denominator contains prime factors other than 2 and 5, then the decimal will be non-terminating and repeating. An irrational number cannot be changed into a terminating or a repeating decimal.

The process of finding roots of an equation by factoring the left side, when the right side equals 0, sometimes gives rise to difficulties; and hence the teacher should be on his guard in checking results. For example, take the equation,

$$(x-1)\left(5+\frac{7}{x-1}\right)=0.$$

The student naturally thinks that 1 is a root of this equation because it makes the first factor zero, without taking into account what happens to the second factor when x approaches the value 1. In beginning work of this type it is well for the teacher to insist, in checking, that the value of x shall be substituted in each factor.

In recent years there has been great emphasis upon the applications of mathematics; and sometimes this is to neglect fundamental principles. In the consideration of financial problems some texts commit serious errors. Here is an example: "What is the rate of income on a 5% bond, whose face value is $1,000 if it is bought for $900?" Clearly, insufficient data is given to find the yield rate. An example from another text is this: "Find the yield rate on a 6% bond bought May 1, 1916, at 113, due May 1, 1935." The solution is this:

"Loss on \$100 bond $=\dfrac{13}{19}=\$.68$ per year.

The interest $=6.00-0.68=\$5.32$, that is, rate $=5.32\%$."

Bond tables give the answer to this as 4.81%. Such loose work as this should not be taught.

In making applications the subject matter should be such as will appeal to the student, and not something beyond his experience. Yield rates on bonds should not be taken up until the student has a fair knowledge of algebra, including geometric series and logarithms.

The teacher should gradually acquire a critical attitude toward his subject. One way to do this is to find out what texts or treatises explain carefully the fundamentals.

The fundamental idea in all our teaching is that mathematics is of great importance as a mode of thought; and we need to keep continually in mind that there is danger of being sidetracked in the making of applications. Naturally we need concrete or vitalizing material, but we must always impress upon the students the importance of the trained mind.

Problem Department

Edited by
E. P. STARKE and N. A. COURT*

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscripts be typewritten with double spacing. Send all communications to EMORY P. STARKE, Rutgers University, New Brunswick, N. J.

SOLUTIONS

No. 411. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Given the angles B and E. Draw a line $ACDF$, parallel to a given line and cutting the sides of angle B in A and C, and those of angle E in D and F so that $AC=DF$.

Solution by "*Nev. R. Mind.*"

Let U, V be the areas of the triangles ABC, DEF; let g and h be their altitudes issued from B and E; and let the line BE meet $ACDF$ in L. Assuming that $AC=DF$, we have:

$$BL : EL = g : h = U : V = \sin A \cdot \sin C/\sin B : \sin D \cdot \sin F/\sin E.$$

Now the six angles involved are known, for the direction of the line $ACDF$ is given. Hence the point L divides the given segment BE in a given ratio, either internally or externally. The parallels to the given line through the two positions of the point L so constructed are two solutions of the proposed problem.

EDITOR'S NOTE: Let m be the parallel to the given line u through the point E, and let $x=ACDF$ be any parallel to u. Through A draw a parallel to DE meeting the line m in E'. If the parallel to EF through E' should happen to pass through C, the line x would be a solution of the problem, for the triangles $AE'C$, DEF would obviously be congruent.

If the parallel to EF through E' does not pass through C, let C' be its trace on the line BC. If P, Q, R are the points at infinity of the lines u, DE, EF, we have, as the line $x = ACDF$ varies,

$$(C\cdots)\overline{\wedge}P(C\cdots)\overline{\wedge}(A\cdots)\overline{\wedge}Q(A\cdots)\overline{\wedge}(E'\cdots)\overline{\wedge}R(E'\cdots)\overline{\wedge}(C'\cdots)$$

The points C, C' thus describe two superposed projective ranges on the fixed line BC. A parallel to the line u through a double element of the two ranges considered constitutes a solution of the problem. There may be two solutions.

No. 465. Proposed by *Paul D. Thomas*, Lucedale, Miss.

The polar of a fixed point M with respect to the conic

$$x^2/(a^2+t)+y^2/(b^2+t)=1 \quad \text{(parameter } t\text{)}$$

meets the conic in P and Q. The perpendicular through M to the polar meets the conic in R and S and the polar in N. Show that

I. The polar of M with respect to the conic envelopes a parabola;
II. P, Q and N trace the same cubic curve passing through M and the foci of the given conic;
III. R and S trace a quartic curve passing through M.

Solution by the *Proposer*.

I. The polar of $M(c,f)$ with respect to the given conic is

$$cx/(a^2+t)+fy/(b^2+t)=1, \tag{1}$$

which may be written

$$t^2+(a^2+b^2-fy-cx)t+a^2b^2-b^2cx-a^2fy=0.$$

The discriminant of this quadratic in t is

$$(cx+fy)^2-2(a^2-b^2)(cx-fy)+(a^2-b^2)^2=0,$$

which is the desired envelope, a parabola since the terms of the second degree form a perfect square.

II. The perpendicular from $M(c,f)$ to (1) is

$$f(x-c)/(b^2+t)-c(y-f)/(a^2+t)=0. \tag{2}$$

N is the intersection of (1) and (2). Thus (1) and (2) together give a parametric representation of the locus of N. The elimination of t gives

$$(cy-fx)(x^2+y^2-cx-fy)=(a^2-b^2)(x-c)(y-f). \tag{3}$$

By direct substitution it is seen that $M(c,f)$ and the foci $((a^2-b^2)^{\frac{1}{2}}, 0)$ lie on (3). Similarly the equation of the conic and (1) are parametric equations for the locus of P and Q. Elimination of t gives again the cubic (3).

III. Parametric equations for the locus of R and S are given by (2) and the equation of the given conic. Elimination of t gives

$$(cy-fx)(fx^3+cy^3-cfx^2-cfy^2)=cf(a^2-b^2)(x-c)(y-f),$$

a quartic which clearly passes through $M(c,f)$.

No. 466. Proposed by *Orval D. Hughes*, student, Colgate University.

Find an approximation to the sum of the reciprocals of all the integers from 1000 to 2000 inclusive.

Solution by the *Proposer*.

Consider the representation in rectangular coordinates of the function $f(x)=1/x$. The first thousand of the one thousand and one reciprocals may be represented by a series of rectangles, each having its base on the X-axis and its upper left corner on the curve $y=1/x$. It will then be seen that

$$\sum_{x=1000}^{1999} \frac{1}{x} > \int_{1000}^{2000} \frac{1}{x}dx = ln2.$$

Similarly, represent the last thousand of the one thousand and one terms of the series by rectangles, each with its base on the X-axis and its upper right corner on the curve. It will then follow that

$$\sum_{x=1001}^{2000} \frac{1}{x} < ln2.$$

If the missing reciprocal is added in each case and the two inequalities are combined, the result is

$$ln2+.0005 < \sum_{x=1000}^{2000} \frac{1}{x} < .001+ln2.$$

Hence the sum of the reciprocals of the integers from 1000 to 2000 inclusive lies between .69364 and .69415.

No. 468. Proposed by *W. E. Byrne*, Virginia Military Institute.

Find $\lim_{x\to 0} \frac{1}{x} \text{arc cos} \frac{\sin x}{x}, \quad 0 \leqq \text{arc cos } y \leqq \pi.$

Solution by the *Proposer*.

If we put $$z = \text{arc cos} \frac{\sin x}{x},$$

we have $$\sin^2 z = 1 - \cos^2 z = 1 - \left(\frac{\sin x}{x}\right)^2$$
$$= 1 - (1 - x^2/6 + \epsilon x^4)^2 = x^2/3 + \lambda x^4,$$

in which ϵ and λ are finite. Then

$$\sin z = \frac{|x|}{\sqrt{3}}[1 + 3\lambda x^2]^{\frac{1}{2}},$$

and $$\lim_{x\to 0} \frac{z}{|x|} = \lim_{x\to 0} \frac{z}{\sin z} \cdot \frac{\sin z}{|x|} = \frac{1}{\sqrt{3}}.$$

This result may be expressed

$$\lim_{x\to +0} \frac{1}{x} \text{arc cos} \frac{\sin x}{x} = \frac{1}{\sqrt{3}}, \quad \lim_{x\to -0} \frac{1}{x} \text{arc cos} \frac{\sin x}{x} = \frac{-1}{\sqrt{3}}.$$

The following fallacious argument appears in Tétrel, *Solutions de Mathématiques Spéciales*, 5ᵉ édition, Paris, no date. Since $(\sin x)/x$ is an even function of x, the expansion of arc cos(sin x/x must be of the form $ax^2 + bx^4 + \cdots$, whence the desired limit is 0. Detection of the fallacy should be a simple exercise.

No. 473. Proposed by *Howard D. Grossman*, New York City.

Prove the obvious generalization of the following relation:

$$\sum_{x=1}^{n} x^5 = \frac{n(n+1)}{6!} \begin{vmatrix} 2 & 0 & 0 & 0 & 1 \\ -1 & 3 & 0 & 0 & n \\ 1 & -3 & 4 & 0 & n^2 \\ -1 & 4 & -6 & 5 & n^3 \\ 1 & -5 & 10 & -10 & n^4 \end{vmatrix},$$

where the portion of the determinant below the principal diagonal is identical with a portion of the Pascal triangle except for the negative

signs in alternate diagonals. The determinant is unchanged in value if all signs are made positive and n is replaced by $n+1$.

Solution by *Gerald B. Huff*, Southern Methodist University, Dallas, Texas.

The relation suggested in the problem may be written

$$\sum_{x=1}^{n} x^k = \frac{n(n+1)}{(k+1)!} \begin{vmatrix} {}_1C_0+1 & 0 & 0 & \cdots\cdots 1 \\ -{}_2C_0 & {}_2C_1+1 & 0 & \cdots\cdots n \\ {}_3C_0 & -{}_3C_1 & {}_3C_2+1 & \cdots\cdots n^2 \\ \cdot & \cdot & \cdot & \cdot \\ (-1)^{k+1}{}_kC_0 & (-1)^k{}_kC_1 & (-1)^{k-1}{}_kC_2 & \cdots\cdots n^{k-1} \end{vmatrix},$$

where ${}_rC_s$ is the binomial coefficient $r!/(r-s)!s!$. It is known that the left member is equal to a polynomial of degree $k+1$ which has $n(n+1)$ as a factor. Hence we set

$$x^k = n(n+1)f_{k-1}(n), \tag{1}$$

$$f_{k-1}(n) = a_0 + a_1 n + a_2 n^2 + \cdots + a_{k-1} n^{k-1}, \tag{2}$$

and proceed as follows:

$$\begin{aligned} n^k = \sum_{x=1}^{n} x^k - \sum_{x=1}^{n-1} x^k &= n(n+1)f_{k-1}(n) - (n-1)nf_{k-1}(n-1) \\ &= n\{[1+1]a_0 + [(n+1)n-(n-1)^2]a_1 + [(n+1)n^2-(n-1)^3]a_2 + \\ &\quad \cdots + [(n+1)n^{k-1}-(n-1)^k]a_{k-1}\}. \end{aligned}$$

$$\begin{aligned} n^{k-1} = [1+1]a_0 &+ [n^2-(n-1)^2+n]a_1 + \cdots \\ &+ [n^k-(n-1)^k+n^{k-1}]a_{k-1}. \end{aligned}$$

$$\begin{aligned} n^{k-1} = a_0\{{}_1C_0+1\} &+ a_1\{-{}_2C_0+[{}_2C_1+1]n\} \\ &+ a_2\{{}_3C_0 - {}_3C_0 n + [{}_3C_2+1]n^2\} + \cdots \\ &+ a_{k-1}\{(-1)^{k+1}{}_kC_0 + (-1)^k{}_kC_1 n + (-1)^{k-1}{}_kC_2 n^2 + \cdots \\ &+ [{}_kC_{k-1}+1]n^{k-1}\}. \end{aligned} \tag{3}$$

If we equate coefficients in this identity in n, we have

$$\begin{aligned} [{}_1C_0+1]a_0 - {}_2C_0a_1 + {}_3C_0a_2 + \cdots + (-1)^{k+1}{}_kC_0a_{k-1} &= 0, \\ [{}_2C_1+1]a_1 - {}_3C_1a_2 + + \cdots + (-1)^k{}_kC_1a_{k-1} &= 0, \\ [{}_3C_2+1]a_2 + \cdots + (-1)^{k-1}{}_kC_2a_{k-1} &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\ [{}_kC_{k-1}+1]a_{k-1} - 1 &= 0. \end{aligned}$$

Elimination of $a_0, a_1, a_2, \cdots, a_{k-1}$ between these equations and (2) yields

$$\begin{vmatrix} {}_1C_0+1 & -{}_2C_0 & {}_3C_0 & \cdots(-1)^{k+1}{}_kC_0 & 0 \\ 0 & {}_2C_1+1 & -{}_3C_1 & \cdots\ (-1)^{k}{}_kC_1 & 0 \\ 0 & 0 & {}_3C_2+1 & \cdots(-1)^{k-1}{}_kC_2 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots\ {}_kC_{k-1}+1 & -1 \\ 1 & n & n^2 & \cdots\ n^{k-1} & -f_{k-1}(n) \end{vmatrix} = 0$$

or

$$(4) \qquad -(k+1)!f_{k-1}(n)+\begin{vmatrix} {}_1C_0+1 & -{}_2C_0 & {}_3C_0 & \cdots & (-1)^{k+1}{}_kC_0 \\ 0 & {}_2C_1+1 & -{}_3C_1 & \cdots & (-1)^{k}{}_kC_1 \\ 0 & 0 & {}_3C_2+1 & \cdots & (-1)^{k-1}{}_kC_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & n & n^2 & \cdots & n^{k-1} \end{vmatrix} = 0.$$

Except for an interchange of rows and columns, this is the desired result.

If we set

$$f_{k-1}(n) = b_0 + b_1(n+1) + b_2(n+1)^2 + \cdots + b_{k-1}(n+1)^{k-1},$$

the work is quite similar to the above except that in the analogue of (3) all the signs are positive. As a result we would get $(k+1)!f_{k-1}(n)$ equal to the determinant in (4) with all the signs positive and n replaced by $n+1$.

PROPOSALS

No. 504. Proposed by *Walter B. Clarke*, San Jose, California.

Construct a triangle such that an altitude, a median and an external bisector of an angle shall be concurrent.

No. 505. Proposed by *Paul D. Thomas*, Sherburne, New York.

Determine the surface generated by a variable circle having for diameter a diameter of a fixed hyperbola, the plane of the circle being perpendicular to the plane of the hyperbola.

No. 506. Proposed by *J. M. Hurt*, Sanatorium, Texas.

Two smooth wires are fitted, one along the (vertical) Y-axis and the other along the first quadrant of the hypocycloid

$$x^{2/3} + y^{2/3} = a^{2/3}.$$

If two beads, one on each wire, are released at the same time from rest at the point $(0,a)$, at what rate will the beads be separating when the bead on the straight wire reaches the origin. Assume gravity is the only force acting.

No. 507. Proposed by *V. Thébault*, Tennie, Sarthe, France.

In a tetrahedron the sum of the squares of the altitudes is not greater than four ninths of the sum of the squares of the edges.

No. 508. Proposed by *E. P. Starke*, Rutgers University.

Find an expression in finite terms for the infinite product

$$\cos x \cdot \cos \frac{x}{2} \cdot \cos \frac{x}{4} \cdot \cos \frac{x}{8} \cdots .$$

No. 509. Proposed by *N. A. Court*, University of Oklahoma.

If P, Q, R, S are the symmetrics of a given point with respect to the centroids of the faces BCD, CDA, DAB, ABC of a tetrahedron $ABCD$, the lines AP, BQ, CR, DS have a point in common.

NOTES

There seems to be a lack of variety of problems of the type

$$\lim_{x\to 0} \; [f(x)]^{g(x)}, \quad \text{where} \quad \lim_{x\to 0} \; f(x)=0, \quad \lim_{x\to 0} \; g(x)=0.$$

Practically all that are given in texts have the answer 1. Can our readers supply some interesting examples for which the answer is not unity?

W. E. BYRNE.

To illustrate the need of care in solving equations graphically, the instructor in Theory of Equations can make good use of such examples as

$$x^4-16x+19=0, \qquad x^4-12x+13=0.$$

The first has two roots between 1.5 and 1.7 while the second has no real roots although its minimum point is only about 0.02 units above the X-axis. The equation

$$2x^4-28x^2+56x-31=0$$

has three real roots in the unit interval $1<x<2$.

Surely many examples of unusual interest or value are known to mathematicians who do not publish books. Perhaps we are willing to share them with our colleagues through the pages of the MAGAZINE.

E. P. S.

Bibliography and Reviews

Edited by
H. A. SIMMONS and P. K. SMITH

The Calculus. By H. J. Ettlinger and M. B. Porter. The Dryden Press, New York, 1942. xiii+317 pages; $3.25.

The book contains a one-year's course in Differential and Integral Calculus as given by the authors at the University of Texas, and it seems sufficiently brief to be covered rather thoroughly in that time.

The first 103 pages are concerned with the Differential Calculus, the development being for the most part in the usual order, ending with a short treatment of partial differentiation. The next 87 pages are concerned with Integral Calculus proper, in which a considerable part of the space is devoted to the definite integral considered from the standpoint of summation. Double and triple integrals are here treated briefly. The next 18 pages comprise a chapter on series and methods of approximation. Taylor's Formula with the remainder as an integral, rather than in the Cauchy or Lagrange form, is developed and used in approximating the values of certain functions. Newton's method is used for finding approximate roots of equations, and Simpson's rule for evaluating approximately some definite integrals. A very brief discussion on convergence and divergence of infinite series ends the chapter.

The next chapter of 14 pages is devoted to an introductory treatment of ordinary differential equations, in which homogeneous, second order, and linear differential equations are treated very briefly. The next 35 pages form a chapter entitled "Functions and Their Graphs". The next chapter consists of 21 pages of supplementary problems, arranged in groups to go with each of the preceding chapters; the next one is a collection of 15 pages of formulas and tables, and the last chapter contains the answers to the exercises in the book.

Almost at the beginning of the book, on page 5, the idea of sequence of real numbers is introduced, and it is used just after the derivative has been defined in finding the equations of the tangent and normal to a plane curve. A rather unusual method is used in developing the derivative of u^n, n being considered at once as a rational number rather than first as a positive integer. Early in the book there is introduced in a short article of about two pages the idea of an integral, considered as an antiderivative, and while it is not given much prominence until Integral Calculus is taken up on page 104, there are a few exercises at intervals that keep the concept before the students. On page 64 in testing for maxima and minima the criterion is given that the "average slope" must change sign in some neighborhood about the given point. It seems regrettable that they do not say simply the "slope", as of course the derivative is used in the test rather than the slope of the secant.

The book stresses the principle of limits throughout, has in addition to the regular exercises a class of starred problems for the better students—about 2500 problems in all of both kinds—and the format of the book is attractive. Those interested in a course in the Calculus covering a rather comprehensive list of subjects, all treated briefly, would do well to look it over.

University of North Carolina. ERNEST L. MACKIE.

Mathematics of Air and Marine Navigation. By A. D. Bradley. American Book Company, New York, 1942. vi+100 pages; $1.00.

This recent text is timed to serve a definite need in war training. Upon covering this short text of seven chapters, the student will be grounded in the basic mathematics of marine and air navigation. The author states that the purpose of the text is "to provide the future navigator with a substantial foundation of mathematical theory, and to acquaint the interested layman with some of the problems encountered in navigating aircraft and surface vessels."

This text is based upon a knowledge of plane trigonometry. The necessary spherical trigonometry for surface and air navigation is developed briefly and simply. A set of five-place tables of logarithms of numbers, of the trigonometric functions, and of haversines is included.

The first chapter deals with "Geometry of the Earth." The earth co-ordinates are discussed; introductory ideas on maps are treated briefly.

Chapter II is devoted to the three types of sailings and to dead reckoning.

Piloting by land marks and radio are treated quite briefly in Chapter III.

"Special Problems of Air Navigation" is the subject of Chapter IV. In this chapter the basic ideas on use of Vectors as regards velocities are brought out. The "windstar", interception, radius of action returning to a fixed, and radius of action returning to a moving, base are discussed.

Spherical trigonometry is the material treated in Chapter V.

In the last two chapters the author develops the fundamentals of celestial navigation. The three common types of celestial coordinates and time are treated in Chapter VI. In Chapter VII the optical principles of a sextant and the corrections to be applied to an observed altitude gotten with the sextant are first discussed. Two methods are then given for determination of latitude. The book closes with the Marcz St. Hilaire method for the determination of latitude and longitude.

This text is a real contribution at this time when navigators must be trained rapidly; when mathematical principles must be acquired by many prospective navigators in so short a time.

A few suggestions may make the text more teachable, or more usable for the self-taught student. First a couple of corrections will be mentioned: (a) "latitude" was used incorrectly for "altitude" at the bottom of page 85; (b) in Figure 52 the line through M, stated to have been parallel to M_2B was not drawn quite parallel to M_2B. The text would be helpful to many if the answers had been given; especially, the self-taught student would be assisted. In Figure 48 a small sphere to represent the earth drawn inside the celestial sphere would be helpful in showing the connection between the Greenwich hour angle and local hour angle and the longitude. The prospective navigator might have been given the idea of sidereal time in this text. The method of determination of longitude by the "time sight"—a basic method—could have been given with little additional space. Finally, the wings of a plane are built at 90 degrees to the axis of the plane. In Figure 27 it is confusing to indicate the track at 90 degrees to the heading.

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